TAMING THE COLOURED MULTIZETAS.

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- Part 1: The flexion structure.
- Part 2: Multizeta irreducibles and perinomal analysis.
- Part 3: Bicoloured multizetas and their satellites.

N.B. Spell *flexion*, not *flection*!

Key words: *uncoloured/bicoloured multizetas* , *flexions* , *bimoulds* , *irreducibles* , *ari/gari* , *biari/bigari* , *swap* , *bialternal/bisymmetral* , *perinomal algebra* , *singulators/singulands/singulates* , *mould amplification* , *satellites*.

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Part 1. The flexion structure.

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Part 2. Multizeta irreducibles and perinomal analysis.

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1 Basic flexions.

Bimoulds $M^\bullet$ have a two-tier indexation $\bullet = \boldsymbol{w} = \binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}$ with u_i's and v_i's that interact in a very special way, through four basic flexions $\rfloor, \lceil$ and $\rceil, \lfloor$. Thus, if $\boldsymbol{w} = \boldsymbol{w'}.\boldsymbol{w''}$

with $\boldsymbol{w'} = \binom{u_1\ ,\ u_2}{v_1\ ,\ v_2}$ *and* $\boldsymbol{w''} = \binom{u_3\ ,\ u_4\ ,\ u_5}{v_3\ ,\ v_4\ ,\ v_5}$, we set:

$$\boldsymbol{w'}\rfloor = \begin{pmatrix} u_1 & , & u_2 \\ v_{1:3} & , & v_{2:3} \end{pmatrix} \qquad \lceil\boldsymbol{w''} = \begin{pmatrix} u_{1,2,3} & , & u_4 & , & u_5 \\ v_3 & , & v_4 & , & v_5 \end{pmatrix}$$

$$\boldsymbol{w'}\rceil = \begin{pmatrix} u_1 & , & u_{2,3,4,5} \\ v_1 & , & v_2 \end{pmatrix} \qquad \lfloor\boldsymbol{w''} = \begin{pmatrix} u_3 & , & u_4 & , & u_5 \\ v_{3:2} & , & v_{4:2} & , & v_{5:2} \end{pmatrix}$$

$$u_{i,j,k\dots} := u_i + u_j + u_k \dots \qquad v_{i:j} := v_i - v_j$$

The products of upper and lower indices remain invariant:

$$\boldsymbol{w} = \boldsymbol{w'}\boldsymbol{w''}\ ,\ \boldsymbol{w}^* = \boldsymbol{w'}\rfloor\,\lceil\boldsymbol{w''}\ ,\ \boldsymbol{w}^{**} = \boldsymbol{w'}\rceil\,\lfloor\boldsymbol{w''} \quad \Rightarrow$$

$$\sum u_i\, v_i \equiv \sum u_i^*\, v_i^* \equiv \sum u_i^{**}\, v_i^{**}$$

$$\sum du_i \wedge dv_i \equiv \sum du_i^* \wedge dv_i^* \equiv \sum du_i^{**} \wedge dv_i^{**}$$

2 Basic flexion operations: the core involution *swap*.

$$B^\bullet = \mathrm{swap}\, A^\bullet$$

$$\Updownarrow$$

$$B^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ v_1 & ,\dots, & v_r \end{smallmatrix}\right)} = A^{\left(\begin{smallmatrix} v_r & ,\dots, & v_{3:4} & , & v_{2:3} & , & v_{1:2} \\ u_{1,\dots,r} & ,\dots, & u_{1,2,3} & , & u_{1,2} & , & u_1 \end{smallmatrix}\right)}$$

Once again, the invariance holds : $\sum_i u_i\, v_i = \sum_i v_{i:i+1}\, u_{1,\dots,i}$

• The *swap* transform ($swap^2 = id$) is as central to flexion theory as the Fourier transform ($\mathcal{F}^4 = id$) is to Analysis. There are even contexts where the two coincide.

• Interesting bimoulds $M^\bullet$ tend to possess a *double symmetry* : one for $M^\bullet$, another for the *swappee* ($swap.M^\bullet$).

3 Basic flexion operations: *ari*, *gari*.

Lie bracket *ari* $\Longrightarrow$ Lie algebra *ARI* :

$$N^\bullet = \mathrm{arit}(B^\bullet)M^\bullet \;\Leftrightarrow\; N^w = \sum^{w=abc} M^{a\lceil c}B^{b\rfloor} - \sum^{w=abc} M^{a\rceil c}B^{\lfloor b}$$

$$\mathrm{ari}(A^\bullet, B^\bullet) := \mathrm{arit}(B^\bullet).A^\bullet - \mathrm{arit}(A^\bullet).B^\bullet + \mathrm{lu}(A^\bullet, B^\bullet)$$

Associative product *gari* $\Longrightarrow$ Lie group *GARI* :

$$N^\bullet = \mathrm{garit}(B^\bullet)M^\bullet \;\Leftrightarrow\; N^w = \sum^{w=\prod a^i b^i c^i} M^{\lceil b^1\rceil..\lceil b^s\rceil} B^{a^1\rfloor}..B^{a^s\rfloor} B_*^{\lfloor c^1}..B_*^{\lfloor c^s}$$

$$\mathrm{gari}(A^\bullet, B^\bullet) := \mathrm{mu}(\mathrm{garit}(B^\bullet).A^\bullet, B^\bullet) \quad (B_*^\bullet := \mathit{invmu}\, B^\bullet)$$

NB: $\mathit{gari}(A^\bullet, B^\bullet)$ is linear in $A^\bullet$, highly non-linear in $B^\bullet$.

Main merits: *ari*/*gari* respect double symmetries.

3^* Basic flexion operations: *ari, gari* (*comments*).

Very loosely speaking, the flexion structure is the sum total of all *interesting operations* that may be constructed from the four afore-mentioned flexions. More specifically: up to isomorphism, there exist exactly seven pairs $\{$*Lie algebra, Lie group*$\}$ obtainable in this way. Of these substructures, two have the added distinction of preserving *double symmetries*. Moreover, when restricted to doubly symmetric bimoulds, these two substructures actually coincide. So we choose to work with the simpler of the two pairs: $\{$*ari, gari*$\}$.

4. Origin of the flexion structure in Analysis.

Singularly perturbed systems (typically, differential systems with a small ϵ in front of the leading derivatives) tend to be divergent-resurgent-resummable in $x = \frac{1}{\epsilon}$, giving rise in the Borel ξ-plane to complex singularities ω constructed, *under application of the flexion combinatorics*, from two quite distinct ingredients:

(i) additive u_i-variables that depend solely on the structure of the equation and its multipliers,

(ii) subtractive v_i-variables that reflect the singularities of the equation's coefficients in the multiplicative plane.

4*. Origin of the flexion structure in Analysis (*comments*).

The corresponding developments, esp. the so-called *scramble transform* and the *tesselation bimould*, may be found in

J.E. *Weighted products and parametric resurgence.* Travaux en Cours, 47 1994.

or again, in a much extended context, in

J.E. *Singularly Perturbed Systems, Coequational Resurgence, and Flexion Operations.* 7 June 2014.

The second paper is accessible on the author's homepage.

5. Origin of the flexion structure in mould algebra.

$$C^\bullet = \mathrm{mu}(A^\bullet, B^\bullet) = A^\bullet \times B^\bullet \Leftrightarrow C^{\boldsymbol{u}} = \sum^{\boldsymbol{u}=\boldsymbol{u}'\boldsymbol{u}''} A^{\boldsymbol{u}'} B^{\boldsymbol{u}''}$$

$$C^\bullet = \mathrm{ko}(A^\bullet, B^\bullet) = A^\bullet \circ B^\bullet \Leftrightarrow C^{\boldsymbol{u}} = \sum_{1 \le s}^{\boldsymbol{u}=\boldsymbol{u}^1..\boldsymbol{u}^s} A^{|\boldsymbol{u}^1|, .., |\boldsymbol{u}^s|} B^{\boldsymbol{u}^1} .. B^{\boldsymbol{u}^s}$$

Moulds of the form $\mathcal{M}^\bullet_A = A^\bullet \times Id^\bullet \times A^\bullet_*$ with $A^\bullet \times A^\bullet_* \equiv \mathbf{1}^\bullet$ are stable under (mould) composition, and the equivalence holds:

$$\{\mathcal{M}^\bullet_C = \mathcal{M}^\bullet_A \circ \mathcal{M}^\bullet_B\} \iff \{C^\bullet = \mathrm{gari}(A^\bullet, B^\bullet)\} \qquad (1)$$

The *ari*-bracket is capable of a similar derivation.
R1: From the $\boldsymbol{u}$- to the $\binom{\boldsymbol{u}}{\boldsymbol{v}}$-indexation: *unique extension*.
R2: (1) establishes the associativity of the *gari*-product.
R3: From moulds to bimoulds, and back.

5*. Origin of the flex. str. in mould algebra (*comments*).
There exists a natural path from the basic, non-inflected mould operations (i.e. mould multiplication *mu* or $\times$ with its Lie bracket *lu*, and mould composition *ko* or $\circ$ with a suitably defined Lie bracket *lo*) to the inflected operations *ari, gari*.
Thus, formula (1) shows how to derive *gari* from *mu* and *ko*.
R1: Strictly speaking, (1) derives *gari* only for $\boldsymbol{u}$-dependent bimoulds, but once a flexion operation is defined on the u_i's, it uniquely extends to the v_i's, and *vice versa*.
R2: The quickest way to check the associativity of *gari* is actually by using that very same formula (1).
On the correspondence *uninflected* $\longrightarrow$ *inflected* (which, incidentally, can be partially reversed), see [E5], §1.

6. The coloured multizetas $wa^\bullet$ and $ze^\bullet$.

- Polylogarithmic integrals. $\left(\alpha_j = 0 \text{ or unit root; } \binom{\alpha_1 \neq 0}{\alpha_s \neq 1}\right)$

$$\underline{\mathrm{wa}}^{\alpha_1,\ldots,\alpha_s} := (-1)^{s_0} \int_0^1 \frac{dt_s}{\alpha_s - t_s} \cdots \int_0^{t_3} \frac{dt_2}{\alpha_2 - t_2} \int_0^{t_2} \frac{dt_1}{\alpha_1 - t_1}$$

- Harmonic sums. $\left(e_j = e^{2\pi i \epsilon_j} = \text{unit root; } s_j \in \mathbb{N}^*;\ \binom{e_1}{s_1} \neq \binom{1}{1}\right)$

$$\underline{\mathrm{ze}}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_r}{s_1\ ,\ldots,\ s_r}} := \sum_{n_1 > \ldots > n_r > 0} n_1^{-s_1} e_1^{-n_1} \ldots n_r^{-s_r} e_r^{-n_r} \quad (e_j = e^{2\pi i \epsilon_j})$$

- Conditional conversion rule (assuming convergence):

$$\underline{\mathrm{ze}}^{\binom{\epsilon_1\ \epsilon_2\ ,\ldots,\ \epsilon_r}{s_1\ s_2\ ,\ldots,\ s_r}} \equiv \underline{\mathrm{wa}}^{e_1 \ldots e_r, 0^{[s_r - 1]}, \ldots, e_1 e_2, 0^{[s_2 - 1]}, e_1, 0^{[s_1 - 1]}}$$

- s = *weight* , r = *length* (or *depth*) , $d := s - r$ = *degree*.

7 Algebraic constraints on the scalar multizetas.

• *First symmetry:* $\underline{wa}^\bullet$ is symmetral, with a unique symmetral extension $\underline{wa}^\bullet \to wa^\bullet$ such that $wa^0 = wa^1 = 0$.

• *Second symmetry:* $\underline{ze}^\bullet$ is symmetrel, with a unique symmetrel extension $\underline{ze}^\bullet \to ze^\bullet$ such that $ze^{\binom{0}{1}} = 0$.

• *Conversion rule:* The conversion formula $\underline{wa}^\bullet \leftrightarrow \underline{ze}^\bullet$ has a non-trivial extension $wa^\bullet \leftrightarrow ze^\bullet$, best expressed in terms of the generating series $zag^\bullet$ and $zig^\bullet$. Cf infra.

• *Colour-consistency:* If $p \in \mathbb{N}$, $\mathbb{Q}_\infty := \mathbb{Q}/\mathbb{Z}$, $\mathbb{Q}_p := (\frac{1}{p}\mathbb{Z})/\mathbb{Z}$

$$\sum_{\tau_j \in \mathbb{Q}_p} \underline{\mathrm{ze}}^{\binom{\epsilon_1+\tau_1\ ,\dots,\ \epsilon_r+\tau_r}{s_1\ \ ,\dots,\ \ s_r}} \equiv p^{-d}\, \underline{\mathrm{ze}}^{\binom{p\,\epsilon_1\ ,\dots,\ p\,\epsilon_r}{s_1\ ,\dots,\ s_r}} \quad \textit{with} \quad d := s - r$$

• *Conjecture:* this set of algebraic constraints is exhaustive.

7^* Alg. constraints on the scalar multizetas (*comments*).

Attached to each of the two encodings $\underline{wa}^\bullet$ and $\underline{ze}^\bullet$ there is a specific *symmetry type*, which amounts to a specific way of multiplying the scalar multizetas. This is the essence of *arithmetical dimorphy* — a phenomenon that extends far beyond the multizeta landscape, but finds there its most striking manifestion.

Dropping the convergence assumption while preserving the symmmetries, i.e. extending $\underline{wa}^\bullet, \underline{ze}^\bullet$ to $wa^\bullet, ze^\bullet$, is a purely formal-algebraic affair, but it comes at the cost of a slight complication in the *conversion rule* and *colour consistency* constraints. The modified constraints are best expressed in terms of the generating functions $zag^\bullet, zig^\bullet$ and of two suitable elements in $centre(GARI)$: see slides 9,10.

8 The generating series/functions $zag^\bullet$ and $zig^\bullet$.

- $$\mathrm{zag}^{\binom{u_1\ ,\dots,\ u_r}{\epsilon_1\ ,\dots,\ \epsilon_r}} := \sum_{1\le s_j} \mathrm{wa}^{e_1,0^{[s_1-1]},\dots,e_r,0^{[s_r-1]}}\, u_1^{s_1-1} u_{1,2}^{s_2-1} \dots u_{1,\dots,r}^{s_r-1} \quad (2)$$
- $$\mathrm{zig}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{v_1\ ,\dots,\ v_r}} := \sum_{1\le s_j} \mathrm{ze}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{s_1\ ,\dots,\ s_r}}\, v_1^{s_1-1} \dots v_r^{s_r-1} \quad (3)$$

Meromorphy of $zag^\bullet$ and $zig^\bullet$. Setting $P(t) := \frac{1}{t}$, we have:

$$\begin{aligned}
\mathrm{zag}^\bullet &= \lim_{k\to} \left(\mathrm{dozag}_k^\bullet \times \mathrm{cozag}_k^\bullet\right)\\
\mathrm{zig}^\bullet &= \lim_{k\to} \left(\mathrm{dozig}_k^\bullet \times \mathrm{cozig}_k^\bullet\right)\\
\mathrm{dozag}^{\binom{u_1\ ,\dots,\ u_r}{\epsilon_1\ ,\dots,\ \epsilon_r}} &= \sum_{1\le m_j\le k}\ \prod_{1\le j\le r} e_j^{-m_j}\, P(m_{1,\dots,j}-u_{1,\dots,j})\\
\mathrm{dozig}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{s_1\ ,\dots,\ \epsilon_r}} &= \sum_{k\ge n_1>..>n_r>0}\ \prod_{1\le j\le r} e_j^{-n_j}\, P(n_j-v_j)
\end{aligned}$$

- $zag^\bullet \in GARI^{as/is}$ = no group, but right action of $GARI^{\underline{as}/\underline{is}}$.

8^* The generating series $zag^\bullet$ and $zig^\bullet$ (*comments*).

There is much to be gained by switching from the scalar multizetas $wa^\bullet, ze^\bullet$ to the generating series $zag^\bullet, zig^\bullet$ as defined by (2)-(3). These generating series, crucially, sum to meromorphic functions, which in turn factor into a dominant part $dozag^\bullet$ or $dozig^\bullet$ that carries '*multivariate simple poles*' (recall that $P(t) := 1/t$), and an elementary (scalar-valued) corrective factor $cozag^\bullet$ or $cozig^\bullet$.

The sets $GARI^{as/as}, GARI^{as/is}$ are no groups, but admit a right action of the groups $GARI^{\underline{as}/\underline{as}}, GARI^{\underline{as}/\underline{is}}$, the only difference being that elements of the groups have length-1 components *even* in w_1: $S^{w_1} \equiv S^{-w_1}$.

9. Algebraic constraints on the generating series.

- *First symmetry:* $zag^\bullet$ symmetral.
- *Second symmetry:* $zig^\bullet$ symmetril.

$$\mathrm{zig}^{\dots,w_i+w_j,\dots} \to \mathrm{zig}^{\left(\begin{smallmatrix}\dots, & u_{i,j} & ,\dots\\ \dots, & v_i & ,\dots\end{smallmatrix}\right)} P(v_{i:j}) + \mathrm{zig}^{\left(\begin{smallmatrix}\dots, & u_{i,j} & ,\dots\\ \dots, & v_j & ,\dots\end{smallmatrix}\right)} P(v_{j:i})$$

- *Conversion rule:* For a well-defined $man^\bullet \in GARI_{centre}$

$$\mathrm{swap.zig}^\bullet = \mathrm{gari}(\mathrm{zag}^\bullet, \mathrm{man}^\bullet) = \mathrm{mu}(\mathrm{zag}^\bullet, \mathrm{man}^\bullet)$$

- *Colour-consistency:* For a well-defined $lag_p^\bullet \in GARI_{centre}$

$$\mu_p\,\mathrm{zag}^\bullet = \mathrm{gari}(\delta_{\mathrm{p}}\,\mathrm{zag}^\bullet, \mathrm{lag}_{\mathrm{p}}^\bullet) = \mathrm{mu}(\delta_{\mathrm{p}}\,\mathrm{zag}^\bullet, \mathrm{lag}_{\mathrm{p}}^\bullet) \quad (\forall p \in \mathbb{N})$$

with $$\mu_p\,\mathrm{zag}^{\left(\begin{smallmatrix}u_1 & ,\dots, & u_r\\ \epsilon_1 & ,\dots, & \epsilon_r\end{smallmatrix}\right)} := p^d \sum_{p\epsilon_j' \equiv p\epsilon_j} \mathrm{zag}^{\left(\begin{smallmatrix}u_1 & ,\dots, & u_r\\ \epsilon_1' & ,\dots, & \epsilon_r'\end{smallmatrix}\right)} (\textit{p-averaging})$$

and $$\delta_p\,\mathrm{zag}^{\left(\begin{smallmatrix}u_1 & ,\dots, & u_r\\ \epsilon_1 & ,\dots, & \epsilon_r\end{smallmatrix}\right)} := p^{-r} \mathrm{zag}^{\left(\begin{smallmatrix}u_1/p & ,\dots, & u_r/p\\ \epsilon_1\,p & ,\dots, & \epsilon_r\,p\end{smallmatrix}\right)} \quad (\textit{p-dilation})$$

- Pairs $\{GARI^{\underline{as}/\underline{is}}, ARI^{\underline{al}/\underline{il}}\}$ and $\{GARI^{\underline{as}/\underline{as}}, ARI^{\underline{al}/\underline{al}}\}$.

10. The centre of GARI.

The elements $ca^\bullet$ of $GARI_{centre}$ are all of the form:

$$\mathrm{ca}^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} \equiv \mathrm{ca}_r \in \mathbb{C} \quad \mathit{if} \quad (v_1, \dots, v_r) = (0, \dots, 0) \ \ (\mathit{else} \equiv 0)$$

and verify for all $Ma^\bullet \in GARI$:

$$\mathrm{gari}(\mathrm{ca}^\bullet, \mathrm{Ma}^\bullet) \equiv \mathrm{gari}(\mathrm{Ma}^\bullet, \mathrm{ca}^\bullet) \equiv \mathrm{mu}(\mathrm{Ma}^\bullet, \mathrm{ca}^\bullet)$$

The central elements $man^\bullet$, $lag_p^\bullet$ on slide 9 correspond to constants man_r, $lag_{p,r}$ so defined:

$$\sum_{1\leq r} \mathrm{man}_r\, t^r \equiv \exp\left(\sum_{2\leq s}(-1)^{s-1}\zeta(s)\frac{t^s}{s}\right)$$

$$\mathrm{lag}_{p,r} := \frac{(-\log p)^r}{r!} = \frac{(-1)^r}{r!}\Big(\sum_{a^p=1\,,\, a\neq 1} \log(1-a)\Big)^r$$

11. Adequation of the flexion structure to multizeta arithmetics.

• Moving from the scalar multizetas $wa^\bullet/ze^\bullet$ to the generating series $zag^\bullet/zig^\bullet$ *compactifies* everything.

• $zag^\bullet/zig^\bullet$ simplify the expression of the double symmetry, conversion rule ('dimorphy'), colour consistency etc.

• *GARI* contains, alone of all competing frameworks, such basic and crucially helpful objects as the bimoulds $pal^\bullet/pil^\bullet$.

• The series $zag^\bullet/zig^\bullet$ can also be viewed as *meromorphic functions* resp. in $\boldsymbol{u}$ or $\boldsymbol{v}$, with *simple multivariate poles*. This makes them ideally suited for disentangling the algebraic identities between multizetas, which seem to be wholly derivable from (iterated) polar identities of the form:

$$\frac{1}{n_1^{s_1}\, n_2^{s_2}} = \sum_{\sigma_1,\sigma_2} \Big(\frac{\alpha^{s_1,s_2}_{\sigma_1,\sigma_2}}{n_1^{\sigma_1}\, n_{1,2}^{\sigma_2}} + \frac{\beta^{s_1,s_2}_{\sigma_1,\sigma_2}}{n_2^{\sigma_1}\, n_{2,1}^{\sigma_2}} \Big) = \sum_{\sigma_1,\sigma_2} \Big(\frac{\gamma^{s_1,s_2}_{\sigma_1,\sigma_2}}{n_1^{\sigma_1}\, n_{2:1}^{\sigma_2}} + \frac{\delta^{s_1,s_2}_{\sigma_1,\sigma_2}}{n_2^{\sigma_1}\, n_{1:2}^{\sigma_2}} \Big)$$

12. Dynamical MZs. Reduction of odd-degree MZs.

• Euler considered MZs of length 2. The general MZs first came up in the late 70s, as *dynamical multizetas*, i.e. as the transcendental ingredients of the analytic invariants attached to local, identity-tangent diffeomorphisms.

• Dichotomy: the *arithmetical MZs*, occuring in the Stokes constants and subject to the two symmetries, *versus* the *dynamical MZs*, occuring in the invariants and subject only to those (weaker) algebraic relations *responsible for making the invariants invariant*.

• *Any uncoloured dynamical (and, a fortiori, arithmetical)* $\zeta(s_1, \ldots, s_r)$ *of odd degree* $d := -r + \sum s_i$ *can, via an explicit algorithm, be expressed in terms of MZs of even degree plus, oddly, the 'odd' odd-degreed* $\zeta(2) = \pi^2/6$.

13. The palindromy formula in $ARI^{\underline{al}/\underline{il}}_{ent.}$.

For any $C \in \mathrm{IHARA} \subset \mathbb{Q}[x_0, x_1]$ *(corresponding to elements of* $ARI^{\underline{al}/\underline{il}}_{polynomial}$*), the right and left decompositions*

$$C = A_0\, x_0 + A_1\, x_1 = x_0\, B_0 + x_1\, B_1 \qquad (A_i, B_i \in \mathbb{Q}[x_0, x_1])$$

yield sums $A_0 + A_1$ *and* $B_0 + B_1$ *that are invariant under the palindromic involution*

$$x_{\epsilon_1}\, x_{\epsilon_2} \ldots x_{\epsilon_s} \longleftrightarrow (-1)^{s-1}\, x_{\epsilon_s} \ldots x_{\epsilon_2}\, x_{\epsilon_1}$$

Proof: follows from the 'senary relations' which express the invariance of C under the operator $pushu := adari(pal^{\bullet})\, push$.

14. Elimination of all weight indices equal to 1.

Every multizeta $\mathrm{ze}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_r}{s_1\ ,\ldots,\ s_r}}$ *can be decomposed into a finite sum (over* $\mathbb{Q}$*) of multizetas with partial weights* $s_i > 1$*.*

The solution relies on explicit formulae (it uses a functional projector) and involves delightful combinatorics.

N.B. The statement applies equally to coloured and uncoloured multizetas. In the case of uncoloureds, it can be bettered: see Francis Brown's theorem on the elimination of all partial weights s_i other than 2 and 3 (for motivic multizetas).

15 The basic polar/trigonometric bisymmetrals.

Set $P(t) := \frac{1}{t}$ and $Q(t) := \frac{\pi}{\tan(\pi\, t)}$. Then there exists

- an $\mathrm{ess.}^{ly}$ unique polar pair $pal^\bullet/pil^\bullet \in GARI^{\mathrm{as}/\mathrm{as}}$ with $pal^{w_1,\dots,w_r}$ r-homogeneous in the $P(u_i)$ and $P(u_{1,..,2\,i})$.

- an $\mathrm{ess.}^{ly}$ unique trigonometric pair $tal^\bullet/til^\bullet \in GARI^{\mathrm{as}/\mathrm{as}}$ with $tal^{w_1,\dots,w_r}$ r-homogeneous in π^2, the $Q(u_i)$ and $Q(u_{1,..,2\,i})$.

These two bisymmetrals $pal^\bullet/pil^\bullet$ and $tal^\bullet/til^\bullet$
(i) admit several equivalent definitions/characterisations,
(ii) possess no end of remarkable properties,
(iii) are key to the understanding of multizetas (thrice over!!!),
(iv) cannot be defined in any of the alternative frameworks.

- $pal^\bullet, tal^\bullet \in GARI^{as/as}$ not $GARI^{\underline{as}/\underline{as}}$ $(pal^{w_1}, tal^{w_1}$ w_1-odd).
- $GARI^{as/as}$ no group, but $GARI^{as/as}.GARI^{\underline{as}/\underline{as}} = GARI^{as/as}$.

16 The double symmetry exchanger *adari*$(pal^\bullet)$.

As multizeta investigators, we are chiefly interested in the double symmetries $\underline{al}/\underline{il}$ and $\underline{as}/\underline{is}$, but we must also resort to the double symmetries $\underline{al}/\underline{al}$ and $\underline{as}/\underline{as}$ which have the signal advantage of being iso-length, i.e. of involving only bimould components of the same length. Hence the need for *double symmetry exchangers*, assembled from the bisymmetral $pal^\bullet$:

$$\begin{array}{ccc} \mathrm{GARI}^{\underline{as}/\underline{as}} & \xrightarrow{\mathrm{adgari}(\mathrm{pal}^\bullet)} & \mathrm{GARI}^{\underline{as}/\underline{is}} \\ \uparrow \mathrm{expari} & & \uparrow \mathrm{expari} \\ \mathrm{ARI}^{\underline{al}/\underline{al}} & \xrightarrow{\mathrm{adari}(\mathrm{pal}^\bullet)} & \mathrm{ARI}^{\underline{al}/\underline{il}} \end{array}$$

and operating through adjoint action:

$$\begin{aligned} \mathrm{adgari}(A^\bullet)\, B^\bullet &:= \mathrm{gari}(A^\bullet, B^\bullet, \mathrm{invgari}\, A^\bullet) \\ \mathrm{adari}(A^\bullet) &:= \mathrm{logari.adgari}(A^\bullet).\mathrm{expari} \end{aligned}$$

Mark here the first intervention of $pal^\bullet/pil^\bullet$.

17. Singulators, singulands, singulates.

- Singulator $slank_r$: linear operator, turns $S^\bullet$ into $\Sigma^\bullet$
- Singuland $S^\bullet$: regular, length-1 bimould (parity opp. to r)
- Singulate $\Sigma^\bullet$: singular bialternal with polarity of order $r-1$

$$\mathrm{slank_r}: \qquad S^\bullet \in \mathrm{BIMU}_{1,regular} \mapsto \Sigma^\bullet \in \mathrm{ARI}^{\underline{\mathrm{al}}/\underline{\mathrm{al}}}_{r,singular}$$

$$\begin{aligned} 2\,\mathrm{slank}_r.S^\bullet &= \mathrm{leng}_r.\mathrm{neginvar}.(\mathrm{adari}(\mathrm{pal}^\bullet))^{-1}.\mathrm{mut}(\mathrm{pal}^\bullet).S^\bullet \\ &= \mathrm{leng}_r.\mathrm{pushinvar}.\mathrm{mut}(\mathrm{neg}.\mathrm{pal}^\bullet).\mathrm{garit}(\mathrm{pal}^\bullet).S^\bullet \end{aligned}$$

$$\textit{with} \quad \begin{array}{ccc} \mathrm{mut}(A^\bullet).M^\bullet & := & \mathrm{mu}(\mathrm{invmu}.A^\bullet, M^\bullet, A^\bullet) \\ \mathrm{neginvar} & := & \mathrm{id}+\mathrm{neg} \\ \mathrm{pushinvar} & := & \sum_{0\le r}(\mathrm{id}+\mathrm{push}+\mathrm{push}^2+\cdots+\mathrm{push}^{\mathrm{r}}).\mathrm{leng_r} \end{array}$$

N.B. Inadequacy of *ari*-composition by u_1^{-2} for correcting bialternal singularities.

Mark the second intervention of $pal^\bullet / pil^\bullet$.

17*. Singulators, singulands, singulates (*comments*).

For the purpose of singularity compensation[1] we must be able to remove, at every second induction step, unwanted singular parts of type $\underline{al}/\underline{al}$. This, however, is easier said than done. It calls for sophisticated operators capable of producing, from regular bimoulds, any given bisymmetral singularity at the origin of the $\boldsymbol{u}$-multiplane.

The operators are the *singulators*.

The regular inputs are the *singulands*.

The singular, bisymmetral outputs are the *singulates*.

Here again, the pair $pal^{\bullet}/pil^{\bullet}$ turns out to be the construction's essential ingredient, in combination with the elementary operators $leng_r$, $neginvar$, $pushinvar$, mut. For a precise definition of these, see [E2].

[1] as used repeatedly on slide **18** to construct elements of $ARI^{\underline{al}/\underline{il}}_{\underline{ent}}$.

18. Symmetry-respecting singularity removal.

$$\begin{array}{lll}
\mathrm{l\o ma}^\bullet\|_r & \in \mathrm{ARI}^{\underline{al}/\underline{il}} & \text{and regular at } 0 \\
\downarrow \mathrm{adari}(\mathrm{pal}^\bullet)^{-1} & & \\
\mathrm{vil\o ma}^\bullet\|_r & \in \mathrm{ARI}^{\underline{al}/\underline{al}} & \text{and singular at } 0 \\
\downarrow \text{trivial extension} & & \\
\mathrm{vil\o ma}^\bullet\|_{r+1} & \in \mathrm{ARI}^{\underline{al}/\underline{al}} & \text{and singular at } 0 \\
 & & (\textit{desingularisation}) \\
\downarrow \mathrm{adari}(\mathrm{pal}^\bullet) & & \text{with correction if } r \text{ even} \\
\mathrm{l\o ma}^\bullet\|_{r+1} & \in \mathrm{ARI}^{\underline{al}/\underline{il}} & \text{and regular at } 0
\end{array}$$

18*. Singularity removal (*comments*).

We are now in a position to construct elements $l\o ma^{\bullet}/l\o mi^{\bullet}$ of $ARI^{\underline{al}/\underline{il}}$ inductively on the *length* r (also known as *depth*). Start from length 1, where the $\underline{al}/\underline{il}$ condition reduces to *parity in* w_1. Assume we have already reached some higher *odd* length r. Apply the double symmetry exchanger $adari(pal^{\bullet})^{-1}$ so as to get into the more congenial environment $ARI^{\underline{al}/\underline{al}}$. Then leave the component of length $r+1$ as it is but add a *suitable singulate*[2] to the component of length $r+2$. Lastly, apply $adari(pal^{\bullet})$ to return to $ARI^{\underline{al}/\underline{il}}$, where $l\o ma^{\bullet}/l\o mi^{\bullet}$ is now *defined and regular at* $\boldsymbol{u} = \mathbf{0}$ up to length $r+2$ inclusively.

So much for the general scheme, of which there exist three main specialisations, denoted by the vowels u, o, a in place of the 'zero-vowel' $\o$.

---

[2] i.e. a singulate that verifies the *desingularisation equations* of **19**.

19. Constructing *løma*$^\bullet$ by desingularisation.

The first and simplest desingularisation occurs at length $r = 3$ with a composite singuland $S_{1,2}^{w_1,w_2}$:

$$\mathrm{slank}_{1,2}.S^\bullet_{1,2} = \mathrm{ari}(\mathrm{slank}_1.S^\bullet_1, \mathrm{slank}_2.S^\bullet_2) \quad \textit{with} \quad S^\bullet_{1,2} = S^\bullet_1 \otimes S^\bullet_2$$

For $S^\bullet_{1,2}$, the desingularisation equation reads:

$$S_{1,2}^{\binom{u_1\ ,\ u_2}{\epsilon_1\ ,\ \epsilon_2}} + S_{1,2}^{\binom{u_2\ ,\ u_{1,2}}{\epsilon_{2:1}\ ,\ \epsilon_1}} - S_{1,2}^{\binom{u_1\ ,\ u_{1,2}}{\epsilon_{1:2}\ ,\ \epsilon_2}} - S_{1,2}^{\binom{u_{1,2}\ ,\ u_2}{\epsilon_1\ ,\ \epsilon_{2:1}}} = \textit{earlier terms}$$

For uncoloureds and with conventional notations, we get:

$$S_{1,2}^{u_1,u_2} + S_{1,2}^{u_2,u_1+u_2} - S_{1,2}^{u_1,u_1+u_2} - S_{1,2}^{u_1+u_2,u_2} = \textit{earlier terms}$$

For the general singuland $\mathrm{S}^{u_1,\dots,u_r}_{r_1,\dots,r_k}$, the desingul. eq. reads:

$$\sum_\sigma \epsilon_\sigma S_{r_1,..,r_k}^{\sigma(u_1,..,u_k)} = \textit{earlier terms} \quad (\sigma \in \mathrm{SL}_k(\mathbb{Z}), \epsilon_r \in \{0, \pm 1\})$$

19. Constructing $l\o ma^\bullet$ by desingularisation (*comments*).

To proceed from length r to length $r+2$ (r odd) in the inductive construction of $l\o ma^\bullet$, composite singulands $S^\bullet_{r_1,\ldots,r_k}$ are required, with $2 \le k \le r+1, 1 \le r_i, \sum r_i = r+2$. The corresponding singulates $\Sigma^\bullet_{r_1,\ldots,r_k}$ are obtained as *ari*-products of the simple singulates $\Sigma^\bullet_{r_i}$ and have polarity of order $2+r-k$ at $\boldsymbol{u}=\mathbf{0}$. The step $r \to r+2$ actually resolves itself into a sub-induction on k, from $k=2$ (polarity of order r) to $k=r+1$ (polarity of order 1).

20. The basic trifactorisation.

We have the π^2-isolating, parity-splitting identity:

$$\mathrm{zag}^\bullet = \mathrm{gari}(\mathrm{zag}^\bullet_{\mathrm{I}}, \mathrm{zag}^\bullet_{\mathrm{II}}, \mathrm{zag}^\bullet_{\mathrm{III}})$$

with $\mathrm{zag}^\bullet_{\mathrm{I}} \in GARI^{as/is}, \mathrm{zag}^\bullet_{\mathrm{II}} \in GARI^{\underline{as}/\underline{is}}_{\mathrm{even}}, \mathrm{zag}^\bullet_{\mathrm{III}} \in GARI^{\underline{as}/\underline{is}}_{\mathrm{odd}}$.

$$\begin{aligned}
\mathrm{zag}^\bullet_{\mathrm{I}} &= \mathrm{gari}(\mathrm{tal}^\bullet, \mathrm{invgari} \,.\, \mathrm{pal}^\bullet, \mathrm{expari} \,.\, \mathrm{røma}^\bullet) \\
\mathrm{zag}^\bullet_{\mathrm{II}} &= \mathrm{expari}\Big(\sum_{k \text{ even}} \rho^{s_1,..,s_k}_{*\mathrm{II}} \,\vec{\mathrm{preari}}(\mathrm{løma}^\bullet_{s_1}, ..., \mathrm{løma}^\bullet_{s_k})\Big) \\
\mathrm{zag}^\bullet_{\mathrm{III}} &= \mathrm{expari}\Big(\sum_{k \text{ odd}} \rho^{s_1,..,s_k}_{*\mathrm{III}} \,\vec{\mathrm{preari}}(\mathrm{løma}^\bullet_{s_1}, ..., \mathrm{løma}^\bullet_{s_k})\Big)
\end{aligned}$$

where $\rho^\bullet_{*\mathrm{II}}$ and $\rho^\bullet_{*\mathrm{III}}$ denote two alternal moulds with values in the set of multizeta irreducibles.

Mark the third consecutive intervention of $pal^\bullet/pil^\bullet$ (and first appearance of $tal^\bullet/til^\bullet$).

20^*. The basic trifactorisation *(comments)*.

In the above formulae, *preari* denotes the pre-Lie product behind *ari*, and *expari* the natural exponential from *ARI* to *GARI*. An alternative expression for $zag_{II}^\bullet$, $zag_{III}^\bullet$ would be

$$\mathrm{zag}_{\mathrm{II}}^\bullet = 1^\bullet + \sum_{k \text{ even}} \rho_{\mathrm{II}}^{s_1,..,s_k} \overrightarrow{\mathrm{preari}}(\mathrm{l\o ma}_{\mathrm{s}_1}^\bullet, ..., \mathrm{l\o ma}_{\mathrm{s}_k}^\bullet)$$

$$\mathrm{zag}_{\mathrm{III}}^\bullet = 1^\bullet + \sum_{k \text{ odd}} \rho_{\mathrm{III}}^{s_1,..,s_k} \overrightarrow{\mathrm{preari}}(\mathrm{l\o ma}_{\mathrm{s}_1}^\bullet, ..., \mathrm{l\o ma}_{\mathrm{s}_k}^\bullet)$$

with two symmetral moulds $\rho_{\mathrm{II}}^\bullet$, $\rho_{\mathrm{III}}^\bullet$ that are none other than the mould-exponentials of the alternal moulds $\rho_{*\mathrm{II}}^\bullet$, $\rho_{*\mathrm{III}}^\bullet$. Note that whereas separating $zag_{III}^\bullet$ from the first two factors is easy (a simple flexion formula takes care of that), disentangling $zag_{II}^\bullet$ from $zag_{I}^\bullet$ is arduous and calls for the construction of an auxiliary bimould *røma*$^\bullet$/*rømi*$^\bullet$ analogous to *løma*$^\bullet$/*lømi*$^\bullet$.

21. Chief difficulties: infinitude.

For any given length r, the *first* resp. *second* symmetry amounts to a set of relations between $A^{\boldsymbol{w}}$ and various $A^{\sigma.\boldsymbol{w}}$ resp. between $A^{\boldsymbol{w}}$ and various $A^{\tau.\boldsymbol{w}}$, where $\sigma \in \mathfrak{S}_r$ and $\tau \in \mathfrak{S}_r^* := swap.\mathfrak{S}_r.swap$. Combining the two forces us to work with the group $< \mathfrak{S}_r, \mathfrak{S}_r^* >$ generated by $\mathfrak{S}_r$ and $\mathfrak{S}_r^*$, which group is *infinite* as soon as $r \geq 3$.
This complicates matters, e.g. by precluding the existence of *functional* projectors of ARI onto $ARI^{\underline{al}/\underline{al}}$ or $ARI^{\underline{al}/\underline{il}}$.

N.B. For $r = 2$, $< \mathfrak{S}_2, \mathfrak{S}_2^* >$ essentially reduces (modulo parity) to the biratio group. This explains why length-2 multizetas are quite elementary and decidedly untypical.

22. Chief difficulties: imbrication.

Meant is the imbrication of all multizetas of weight less than s, irrespective of length r or degree d.

- *Uncoloured multizetas.* The *construction* of a generating system $\{l\o ma^{\bullet}_s, s = 3, 5, 7...\}$ of $ARI^{\underline{al}/\underline{il}}$ can be carried out in accordance with the (r, d)-filtration (explain), but the *decomposition* of an element of $ARI^{\underline{al}/\underline{il}}$ into multibrackets of $l\o ma^{\bullet}_s$ cannot (clue: relations between the length-1 bialternals). The solution lies in *perinomal analysis*.

- *Bicoloured multizetas.* The *decomposition* of an element of $ARI^{\underline{al}/\underline{il}}$ into multibrackets can proceed in accordance with the (r, d)-filtration, given any system of generators $\{l\o ma^{\bullet}_s, s = 1, 3, 5...\}$, but the *construction* of such a system cannot (explain). The solution lies in *satellisation*.

23 Enforcing rigidity. Perinomal analysis.

Whereas the length-1 elementary bimoulds $\lambda^{\bullet}_{2\,d}$ with $\lambda^{w_1}_{2\,d} := u_1^{2\,d}$ *are not ari*-free and *do not* generate all polynomial bialternals, due to relations like $ari(\lambda^{\bullet}_2, \lambda^{\bullet}_8) - 3\, ari(\lambda^{\bullet}_4, \lambda^{\bullet}_6) \equiv 0$, the length-1 elementary bimoulds $\xi^{\bullet}_n$ with $\xi^{w_1}_n := P(u_1 - n) - P(u_1 + n)$ freely generate, under the *ari* bracket, the algebra of all *eupolar bialternals* $\Xi^{\bullet}_{\boldsymbol{n}}$, i.e. of all bialternals of type

$$\Xi^{w_1,\ldots,w_r}_{n_1,\ldots,n_r} := \sum_{1 \leq k \leq \frac{(2\,r)!}{r!(r+1)!}}^{\epsilon_l \in \{\pm\}} \prod_{1 \leq l \leq r} P\Big(\sum_{j=j^*_{k,l}}^{j=j^{**}_{k,l}} (u_j + \epsilon_j\, n_j)\Big)$$

For a precise description of *eupolar* bimoulds, see [E2] or [E3].

24 The perinomal realisation $luma^{\bullet}$.

By replacing the polynomial singulands $S^{\bullet}_{\boldsymbol{r}}$ by polar singulands and taking their residues $R^{\bullet}_{\boldsymbol{r}}$ as new unknowns:

$$S^{u_1,\dots,u_r}_{r_1,\dots,r_k} = \sum_{n_i} R^{n_1,\dots,n_r}_{r_1,\dots,r_k} \, P(u_1+n_1)\dots P(u_k+n_k)$$

we move from under-determined, multi-solution systems

$$\sum_{\sigma} \epsilon_{\sigma} \, S^{\sigma(u_1,..,u_k)}_{r_1,..,r_k} = \textit{earlier terms} \qquad \big(u_i \in \mathbb{C}, \sigma \in \mathrm{SL}_k(\mathbb{Z})\big)$$

to well-determined, one-solution systems

$$\sum_{\sigma} \eta_{\sigma} \, R^{\sigma(n_1,..,n_k)}_{r_1,..,r_k} = \textit{earlier terms} \qquad \big(n_i \in \mathbb{Z}, \sigma \in \mathrm{SL}_k(\mathbb{Z})\big).$$

• The new singulands $S^{\bullet}_{\boldsymbol{r}}$ are just 'polar' ; it is the corresponding singulates $\Sigma^{\bullet}_{\boldsymbol{r}}$ that are 'eupolar' .

• We then expand the meromorphic-valued bimould $luma^{\bullet}$ as a series $\sum_s luma^{\bullet}_s$ of homogeneous polynomial-valued bimoulds.

25. Perinomal reduction of uncoloureds.

The prodedure yields well-defined expansions ($s_i \in \{3, 5, 7...\}$)

$$\begin{aligned}
\mathrm{zag}^\bullet_{\mathrm{II/III}} &= \mathrm{expari}\Big(\sum \rho^{s_1,..,s_k}_{*\mathrm{II/III}} \,.\, \vec{\mathrm{preari}}(\mathrm{luma}^\bullet_{\mathrm{s}_1}, ..., \mathrm{luma}^\bullet_{\mathrm{s}_k})\Big)\\
&= 1^\bullet + \sum \rho^{s_1,..,s_k}_{\mathrm{II/III}} \,.\, \vec{\mathrm{preari}}(\mathrm{luma}^\bullet_{\mathrm{s}_1}, ..., \mathrm{luma}^\bullet_{\mathrm{s}_k})\\
\mathrm{zig}^\bullet_{\mathrm{II/III}} &= \mathrm{expira}\Big(\sum \rho^{s_1,..,s_k}_{*\mathrm{II/III}} \,.\, \vec{\mathrm{preira}}(\mathrm{lumi}^\bullet_{\mathrm{s}_1}, ..., \mathrm{lumi}^\bullet_{\mathrm{s}_k})\Big)\\
&= 1^\bullet + \sum \rho^{s_1,..,s_k}_{\mathrm{II/III}} \,.\, \vec{\mathrm{preira}}(\mathrm{lumi}^\bullet_{\mathrm{s}_1}, ..., \mathrm{lumi}^\bullet_{\mathrm{s}_k})
\end{aligned}$$

which in turn, after Taylor expansion in the $\boldsymbol{u}$- resp. $\boldsymbol{v}$-variables, lead to the so-called *perinomal decomposition of multizetas into irreducibles* (with a minor transcendental contribution from *luma*$^\bullet$/*lumi*$^\bullet$ from depth 4 onwards). Moreover, we have explicit expansions for the irreducibles: $\rho^{s_1,\ldots,s_r}_{II/III} = \sum_{1\le n_i} \theta^{n_1,..,n_r}_{II/III}\; n_1^{-s_1} \ldots n_s^{-s_r}\, \rho^{s_1,\ldots,s_r}_{II/III}$ ($\theta^\bullet_{II/III}$ is $\mathbb{Q}$-valued). We construct $\mathrm{zag}^\bullet_I/\mathrm{zig}^\bullet_I$ from *ruma*$^\bullet$/*rumi*$^\bullet$ along the same lines. Remarkably, the lone irreducible $\zeta(2) = \pi^2/6$ causes as much trouble as all other irreducibles taken together!

26 The arithmetical realisations *loma*$^\bullet$, *lama*$^\bullet$.

One may also stick with the polynomial singulands $S_{\boldsymbol{r}}^\bullet$ and enforce uniqueness by adding constraints that keep the denominators *arithmetically simple*. There are two options:

- *lama*$^\bullet$: rather lax constraints but optimal denominators.
- *loma*$^\bullet$: stricter constr.[3], fewer coeffs, slightly subopt. denom.

Thus, for $\boldsymbol{r}=(1,2)$, take $Sa_{1,2}^\bullet$ and $So_{1,2}^\bullet$ $\mathrm{resp}^{\mathrm{ly}}$ of the form:

$$\mathrm{Sa}_{1,2}^{\mathrm{u}_1,\mathrm{u}_2} = \sum_{1\leq\delta\leq[\frac{s-1}{2}]-[\frac{s+1}{6}]} \mathrm{ca}_{2\delta}\,.\,u_1^{2\delta}u_2^{s-2\delta-2}$$

$$\mathrm{So}_{1,2}^{\mathrm{u}_1,\mathrm{u}_2} = \sum_{1\leq\delta\leq[\frac{s-3}{6}]} \mathrm{co}_{2\delta}\,.\,u_1^2u_2\,.\,(u_1^{2\delta}u_2^{s-2\delta-5}+u_2^{2\delta}u_1^{s-2\delta-5})$$

The largest prime in the denominators is $\leq[\frac{s}{3}]$ resp. $\leq[\frac{2s-5}{3}]$.

[3] the *stricter constraints* for $So^\bullet$ mimick the *a priori* symmetries of the perinomal singulands $Su^\bullet$, such as $u_2\,Su_{1,2}^{u_1,u_2}\equiv u_1\,Su_{1,2}^{u_2,u_1}$.

27 Some tantalising arithmetical riddles.

When applied to the *'arithmetical singulands'* $Sa^{\bullet}_{1,2}$, $So^{\bullet}_{1,2}$, the general desingularisation equation

$$S^{u_1,u_2}_{1,2} + S^{u_2,u_1+u_2}_{1,2} - S^{u_1,u_1+u_2}_{1,2} - S^{u_1+u_2,u_2}_{1,2} = \textit{earlier terms}$$

produces in the *denominators* of all the coefficients $ca_{2\delta}$ and $co_{2\delta}$ – and, even more unaccountably, in the *numerators* of some of them – *explicitely describable strings of prime numbers* (which *do not* originate in the "*earlier terms*"!). This generation of prime numbers almost *ex nihilo* is rather unparalleled. It persists, moreover, for the higher order singulands $Sa^{\bullet}_{r_1,..,r_k}$ and $So^{\bullet}_{r_1,..,r_k}$.

28. Pausing midway to take stock.

• We pointed at the outset to the double curse of
(i) *infinitude* (of the underlying group $<\mathfrak{S}_r, \mathfrak{S}_r^* >$) and
(ii) *imbrication* (of all multizetas of weight $\leq s$).

• In the case of *uncoloured multizetas*, we showed how to conquer the curse by imposing polar rigidity, leading to the *perinomal decomposition* of uncoloureds into irreducibles.

• We shall now deal with the *coloured, esp. bicoloured multizetas*, and sketch for them a quite distinct way of defeating the curse, again leading to a lot of fascinating new structure (*satellisation*).

29. Taming the bicoloureds: overall scheme.

Road map: for s fixed, reduce the plethora of data and restore a workable (r,d)-filtration.

$$
\begin{array}{ccc|ccc|c}
\mathrm{ARI}^{\underline{al}/\underline{il}}_{\mathit{bicoloured}} & | & \mathrm{zag}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ \epsilon_1 & ,\dots, & \epsilon_r \end{smallmatrix}\right)} & \begin{smallmatrix} r+d=s \\ \epsilon_i\in\{0,\frac{1}{2}\} \end{smallmatrix} & & \\
(\uparrow)\downarrow \ \mathit{restriction} & | & (\uparrow)\downarrow & & & \\
\mathrm{ARI}^{\underline{al}/\underline{il}}_{\mathit{extremal}} & | & \mathrm{zag}^{\left(\begin{smallmatrix} 0 & ,\dots, & 0 \\ \epsilon_1 & ,\dots, & \epsilon_s \end{smallmatrix}\right)} & d=0,r=s & \| & \mathrm{ARI}^{\mathit{al}}_{\mathit{extremal}} \\
(\uparrow)\downarrow_{\mathit{satellisation}} & | & (\uparrow)\downarrow & \mathit{amplification} & \| & (\uparrow)\downarrow_{\mathit{satellisation}} \\
\mathrm{BIARI}^{\underline{al}/\underline{il}^*}_{\mathit{uncoloured}} & | & \mathrm{sazag}_j^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ 0 & ,\dots, & 0 \end{smallmatrix}\right)} & \begin{smallmatrix} r+d=s \\ j\in\{0,1\} \end{smallmatrix} & \| & \mathrm{BIARI}^{\mathit{al}}_{\mathit{extremal}} \\
\downarrow_{\mathit{specialisation}} & | & \downarrow & & \| & \downarrow_{\mathit{specialisation}}. \\
\mathrm{ARI}^{\underline{al}/\underline{il}}_{\mathit{uncoloured}} & | & \mathrm{zag}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ 0 & ,\dots, & 0 \end{smallmatrix}\right)} & r+d=s & \| & \mathrm{ARI}^{\mathit{al}}_{\mathit{uncoloured}}
\end{array}
$$

29*. Taming the bicoloureds: overall scheme *(comments)*.

The first step (data reduction) keeps the colours ϵ_i but retains only the partial weights $s_i = 1$. In terms of generating series, this means restricting $zag^{\binom{u}{\epsilon}}$ to $zag^{\binom{0}{\epsilon}}$. Surprisingly, such massive pruning entails no loss of information, only a partial occultation of it.

The second step (data re-ordering) replaces $zag^{\binom{0}{\epsilon}}$ by a pair of *colour-free satellites* $sazag_j^{\binom{u}{0}}(j = 0, 1)$ obtained by *mould amplification*. It then transports the *ari, gari* action to such pairs, resulting in operations *biari, bigari* that respect the r-filtration by length.

The third step (data recovery) is about retrieving the full $zag^{\binom{u}{\epsilon}}$ from the satellites. This is particularly easy for the uncoloured part $zag^{\binom{u}{0}}$, where it ultimaltely amounts to a colour-to-degree transfer $zag^{\binom{0}{\epsilon}} \to zag^{\binom{u}{0}}$.

29**. Taming the bicoloureds: overall scheme *(comments)*.

Remark 1: Although the three steps make most sense when applied to $ARI^{\underline{al}/\underline{il}}_{bicoloured}$, the steps 2 and 3 extend to $ARI^{\underline{al}/\cdot}_{bicoloured}$ and should *first* be studied in that context, without the unnecessary assumption $(\underline{\cdot}/\underline{il})$.

Remark 2: Step 2 relies on mould amplification. It simply re-orders the data and re-shapes all flexion operations, which henceforth act on satellite couples and acquire the prefix *bi*. Relative to the extremal algebra $ARI^{\underline{al}/\underline{il}}_{extremal}$, step 2 doesn't bring about any data compression, but it instaures a precious r-filtration that was clearly absent from $ARI^{\underline{al}/\underline{il}}_{extremal}$.

Remark 3: The whole three-stepped construction also extends, *mutatis plurimis mutandis* and with less compelling usefulness, to all multicoloured (not just bicoloured) multizetas.

30. The extremal algebra: no information loss.

Def. $A^\bullet$ is dubbed *weakly alternal* if it verifies all alternality relations $\sum_{\boldsymbol{w}\in\mathrm{sha}(\boldsymbol{w}',\boldsymbol{w}'')} A^{\boldsymbol{w}} \equiv 0$ with $\boldsymbol{w}'$ of length 1 and $\boldsymbol{w}''$ of any length. The same applies for *weakly alternil*.

L 1: In a double symmetry, either symmetry may be weakened:

$$\{\mathrm{al}/\mathrm{al}\} \Longleftrightarrow \{\mathrm{al}^{\mathrm{weak}}/\mathrm{al}\} \Longleftrightarrow \{\mathrm{al}/\mathrm{al}^{\mathrm{weak}}\} \not\Leftarrow \{\mathrm{al}^{\mathrm{weak}}/\mathrm{al}^{\mathrm{weak}}\}$$
$$\{\mathrm{al}/\mathrm{il}\} \Longleftrightarrow \{\mathrm{al}^{\mathrm{weak}}/\mathrm{il}\} \Longleftrightarrow \{\mathrm{al}/\mathrm{il}^{\mathrm{weak}}\} \not\Leftarrow \{\mathrm{al}^{\mathrm{weak}}/\mathrm{il}^{\mathrm{weak}}\}$$

L 2: If $A^{\binom{u_1\ ,\dots,\ u_r}{\epsilon_1\ ,\dots,\ \epsilon_r}} \in \mathrm{ARI}_{d,r}^{al/il^{\mathrm{weak}}_{(1,r)}}$ and is colour consistent, then $A^{\binom{u_1\ ,\dots,\ u_r}{\epsilon_1\ ,\dots,\ \epsilon_r}} \equiv 0$.

It follows that the extremal component $A^{\binom{0\ ,\dots,\ 0}{\epsilon_1\ ,\dots,\ \epsilon_s}} \in \mathrm{ARI}^{\underline{al}/\underline{il}}_{extremal}$ successively determines all components $A^{\binom{u_1\ ,\dots,\ u_r}{\epsilon_1\ ,\dots,\ \epsilon_r}} \in \mathrm{ARI}^{\underline{al}/\underline{il}}_{d,r}$ of higher degree d and lesser length r $(d+r\equiv s)$.

30^*. The extremal algebra *(comments)*.

The *colour consistency* assumption is essential. Without it, Lemma 2 fails, and there is no retrieval of information. Indeed, for any two elements $A_1^\bullet$, $A_2^\bullet$ in $ARI^{\underline{al}/\underline{il}}_{bicoloured}$, of weights $s_1 \neq s_2$, set $decA_j^{\binom{u}{\epsilon}} := A_j^{\binom{u}{0}}\ \forall \epsilon$. Then $ari(decA_1^\bullet, decA_2^\bullet)$ is $(\underline{al}/\underline{il})$ but not colour-consistent, and since its trace in the extremal algebra is nil, it cannot be reconstituted therefrom.

31 The extremal algebra: the second symmetry.

Take $A^\bullet \in ARI^{\underline{al}/\underline{il}}_{bicol^d}$, set $\lambda_{d,\epsilon_0}^{\binom{u_1}{\epsilon_1}} = \left\{\begin{smallmatrix} u_1^d & if\ \epsilon_1=\epsilon_0 \\ 0 & otherwise \end{smallmatrix}\right.$, and expand $A^\bullet$:

$$A^\bullet = \sum b^{\epsilon_1,\ldots,\epsilon_s}\ \vec{\mathrm{lu}}(\lambda^\bullet_{0,\epsilon_1}, \lambda^\bullet_{0,\epsilon_2}, \ldots, \lambda^\bullet_{0,\epsilon_s})\ \ if\ \ length(\bullet) = s$$

$$A^\bullet = \sum c^{\epsilon_1,\ldots,\epsilon_{s-1}}\ \vec{\mathrm{lu}}(\lambda^\bullet_{1,\epsilon_1}, \lambda^\bullet_{0,\epsilon_2}, \ldots, \lambda^\bullet_{0,\epsilon_{s-1}})\ \ if\ \ length(\bullet) = s-1$$

$$(\mathrm{swap.Wil.swap}\ A)^{\binom{0\ ,\ldots,\ 0}{\epsilon_1\ ,\ldots,\ \epsilon_s}} = \sum^{*} A^{w^*} + \sum^{**} A^{w^{**}} P(u_{**}) \qquad (4)$$

For $(\epsilon_1, \ldots, \epsilon_s)$ ending with $\epsilon_s = 0$ resp. $\epsilon_s = \frac{1}{2}$, (4) yields:

$$0 = \sum H^{\epsilon_1,\ldots,\epsilon_{s-1}}_{\epsilon'_1,\ldots,\epsilon'_s} b^{\epsilon'_1,\ldots,\epsilon'_s} + c^{\epsilon_1,\ldots,\epsilon_{s-1}} \qquad (H^\bullet_\bullet, K^\bullet_\bullet, L^\bullet_\bullet \in \mathbb{Z}) \qquad (5)$$

$$0 = \sum K^{\epsilon_1,\ldots,\epsilon_{s-1}}_{\epsilon'_1,\ldots,\epsilon'_s} b^{\epsilon'_1,\ldots,\epsilon'_s} + \sum L^{\epsilon_1,\ldots,\epsilon_{s-1}}_{\epsilon''_1,\ldots,\epsilon''_s} c^{\epsilon''_1,\ldots,\epsilon''_{s-1}} \qquad (6)$$

Eliminating $c^\bullet$, we get 2^{s-1} structure constraints on $ARI^{\underline{al}/\underline{il}}$:

$$0 = \sum R^{\epsilon_1,\ldots,\epsilon_{s-1}}_{\epsilon'_1,\ldots,\epsilon'_s} b^{\epsilon'_1,\ldots,\epsilon'_s} \qquad (R^\bullet_\bullet \in \mathbb{Z}) \qquad (7)$$

31^* The second symmetry *(comments)*.

For elements of the extremal algebra $ARI^{\underline{al}/\underline{il}}_{extremal}$, we always have $d = 0$, hence $r = s$. Since all alternility relations commingle components of various lengths, there seems to be no way of expressing them *within* $ARI^{\underline{al}/\underline{il}}_{extremal}$. Weak alternility, however, involves only two consecutive lengths, e.g. $r = s$, $r = s - 1$, and that too in such a way as to permit the elimination of the *external* data $c^{\epsilon_1,\dots,\epsilon_{s-1}}$ between (5) and (6), leading to the constraints (7), which are purely *internal* to the extremal algebra.

R1: In (4), *Wil* simply denotes the linearisation (resp. annihilation) operator for *symmetril* (resp. *alternil*) bimoulds, relative to the sequence splitting $(w_1, .., w_r) \to (w_1)(w_2, .., w_r)$.

R2 : We must take *all* the multibrackets $\vec{lu}(\lambda^\bullet_{1,\epsilon_1}, \dots, \lambda^\bullet_{0,\epsilon_{s-1}})$ to get a basis for the degree-1 alternals, but only *some* of the $\vec{lu}(\lambda^\bullet_{1,\epsilon_0}, \dots, \lambda^\bullet_{0,\epsilon_s})$ to generate the degree-0 alternals.

32 Mould amplification.

We already used mould amplification to go from $wa^\bullet$ to $zag^\bullet$. We shall use it again to construct the *satellites* of bicoloureds. Here are the basic facts:

Mould amplification amp_{ω_*}

(i) singles out a special index ω_*,

(ii) adds a new indexation layer (here, the u_i indices),

(iii) preserves (simple) symmetries.

$$(\mathrm{amp}_{\omega_*} M)^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ \omega_1 & ,\dots, & \omega_r \end{smallmatrix}\right)} := \sum_{0 \le n_r} M^{\omega_1, \omega_*^{[n_1]}, \dots, \omega_r, \omega_*^{[n_r]}} \, u_1^{n_1} \, u_{1,2}^{n_2} \dots u_{1,..,r}^{n_r}$$

If $\mathrm{M}^\bullet$ *is alternal or symmetral, so is* $\mathrm{amp}_{\omega_*} M^\bullet$.

N.B. $\omega_*^{[n]} := \overbrace{\omega_*, .., \omega_*}^{n \text{ times}}$ and $u_{1,..,j} := u_1 + .. + u_j$ as usual.

33. The satellites $sazag_0^\bullet, sazag_1^\bullet$ and $sal\o ma_0^\bullet, sal\o ma_1^\bullet$.

$$\mathrm{sazag}_0^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ 0 & ,\dots, & 0 \end{smallmatrix}\right)} := \sum_{0\le n_r} \mathrm{zag}^{\left(\begin{smallmatrix} 0 & \overbrace{,0\ ,\dots,\ 0,}^{\leftarrow n_1 \rightarrow} & ,\dots, & 0 & \overbrace{,0\ ,\dots,\ 0}^{\leftarrow n_r \rightarrow} \\ 1/2 & ,0\ ,\dots,\ 0, & ,\dots, & 1/2 & ,0\ ,\dots,\ 0 \end{smallmatrix}\right)} u_1^{n_1} u_{1,2}^{n_2} \dots u_{1,..,r}^{n_r}$$

$$\mathrm{sazag}_1^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ 0 & ,\dots, & 0 \end{smallmatrix}\right)} := \sum_{0\le n_r} \mathrm{zag}^{\left(\begin{smallmatrix} 0, & \overbrace{0\ ,\dots,\ 0,}^{\leftarrow n_1 \rightarrow} & ,\dots, 0, & \overbrace{0\ ,\dots,\ 0}^{\leftarrow n_r \rightarrow} \\ 0, & 1/2\ ,\dots,\ 1/2, & ,\dots, 0, & 1/2\ ,\dots,\ 1/2 \end{smallmatrix}\right)} u_1^{n_1} u_{1,2}^{n_2} \dots u_{1,..,r}^{n_r}$$

The satellites $sazag_0^\bullet, sazag_1^\bullet$ inherit symmetralness.

$$\mathrm{sal\o ma}_0^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ 0 & ,\dots, & 0 \end{smallmatrix}\right)} := \sum_{0\le n_r} \mathrm{l\o ma}^{\left(\begin{smallmatrix} 0 & \overbrace{,0\ ,\dots,\ 0,}^{\leftarrow n_1 \rightarrow} & ,\dots, & 0 & \overbrace{,0\ ,\dots,\ 0}^{\leftarrow n_r \rightarrow} \\ 1/2 & ,0\ ,\dots,\ 0, & ,\dots, & 1/2 & ,0\ ,\dots,\ 0 \end{smallmatrix}\right)} u_1^{n_1} u_{1,2}^{n_2} \dots u_{1,..,r}^{n_r}$$

$$\mathrm{sal\o ma}_1^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ 0 & ,\dots, & 0 \end{smallmatrix}\right)} := \sum_{0\le n_r} \mathrm{l\o ma}^{\left(\begin{smallmatrix} 0, & \overbrace{0\ ,\dots,\ 0,}^{\leftarrow n_1 \rightarrow} & ,\dots, 0, & \overbrace{0\ ,\dots,\ 0}^{\leftarrow n_r \rightarrow} \\ 0, & 1/2\ ,\dots,\ 1/2, & ,\dots, 0, & 1/2\ ,\dots,\ 1/2 \end{smallmatrix}\right)} u_1^{n_1} u_{1,2}^{n_2} \dots u_{1,..,r}^{n_r}$$

The satellites $sal\o ma_0^\bullet, sal\o ma_1^\bullet$ inherit alternalness.

33*. The satellites *(comments)*.

Each of the two sets of satellites, whether it be the one consisting of 0-indexed, $amp_{\binom{0}{0}}$-generated satellites, or the one with 1-indexed,$amp_{\binom{0}{1/2}}$-generated satellites, explicitely carries the whole information present in the extremal algebra, and either set can be deduced from the other, but under a clumsy correspondence that exchanges *length* and *degree*. Worse still, if we were to retain only one set of satellites (say, the one with index 0), there would be no *natural* way of extending the flexion operations to that set. So we find ourselves in one of those not infrequent instances where a slight data redundancy is unavoidable.

34. Recovering $\mathit{l\o ma}^\bullet$ from $\mathit{sal\o ma}_0^\bullet$ and $\mathit{sal\o ma}_1^\bullet$

For $\mathit{l\o ma}^\bullet$ in $\mathrm{ARI}^{\underline{al}/\underline{il}}$ and $\mathit{l\o ma}^{\boldsymbol{w_1}} = 0$:

$$\begin{aligned} \mathit{l\o ma}^\bullet{}_{\mathit{bicoloured}} &\rightarrow (\mathit{sal\o ma}_0^\bullet, \mathit{sal\o ma}_1^\bullet)_{\mathit{uncoloured}} \\ \mathit{l\o ma}^\bullet{}_{\mathit{uncoloured}} &\equiv \mathit{neg}.\mathit{sal\o ma}_1^\bullet - \mathit{neg}.\mathit{sal\o ma}_0^\bullet \end{aligned}$$

For $\mathit{s\o ma}^\bullet$ in $\mathrm{GARI}^{\underline{as}/\underline{is}}$ and $\mathit{s\o ma}^{\boldsymbol{w_1}} = 0$:

$$\begin{aligned} \mathit{s\o ma}^\bullet{}_{\mathit{bicoloured}} &\rightarrow (\mathit{sas\o ma}_0^\bullet, \mathit{sas\o ma}_1^\bullet)_{\mathit{uncoloured}} \\ \mathit{s\o ma}^\bullet{}_{\mathit{uncoloured}} &\equiv (\mathit{neg}.\mathit{sas\o ma}_0^\bullet)^{-1} \times \mathit{neg}.\mathit{sas\o ma}_1^\bullet \end{aligned}$$

Interpretation: The extremal algebra carries the full information and so does *each* satellite. However, *explicitely accessing* the occulted information is specially easy for the *uncoloured part*, provided we use *both* satellites.

N.B. $\mathit{neg}.A^{w_1,\dots,w_r} := A^{-w_1,\dots,-w_r}$.

35. The satellite algebra structure (for alternals).

$$
\begin{array}{ccccc}
C^\bullet & \xleftarrow{\mathrm{lu}} & (A^\bullet, B^\bullet) & \xrightarrow{\mathrm{ari}} & D^\bullet \\
\mathrm{sa} \downarrow & & \mathrm{sa} \downarrow \mathrm{sa} & & \downarrow \mathrm{sa} \\
\{C_0^\bullet, C_1^\bullet\} & \xleftarrow{\mathrm{bilu}} & (\{A_0^\bullet, A_1^\bullet\}, \{B_0^\bullet, B_1^\bullet\}) & \xrightarrow{\mathrm{biari}} & \{D_0^\bullet, D_1^\bullet\}
\end{array}
$$

In the absence of length-1 components:

$$
\begin{aligned}
\mathrm{bilu} : \quad & C_0^\bullet = \mathrm{lu}(A_0^\bullet, B_0^\bullet) \ \ , \ \ C_1^\bullet = \mathrm{lu}(A_1^\bullet, B_1^\bullet) \\
\mathrm{biari} : \quad & D_0^\bullet = -\mathrm{ari}(A_0^\bullet, B_0^\bullet) + \mathrm{arit}(B_1^\bullet)A_0^\bullet - \mathrm{arit}(A_1^\bullet)B_0^\bullet \\
& D_1^\bullet = +\mathrm{ari}(A_1^\bullet, B_1^\bullet) - \mathrm{arit}(B_0^\bullet)A_1^\bullet - \mathrm{arit}(A_0^\bullet)B_1^\bullet \\
& \qquad -\mathrm{lu}(A_0^\bullet, B_1^\bullet) - \mathrm{lu}(A_1^\bullet, B_0^\bullet)
\end{aligned}
$$

Remark: $D_1^\bullet - D_0^\bullet = \mathrm{ari}(A_1^\bullet - A_0^\bullet \, , \, B_1^\bullet - B_0^\bullet)$

35*. The satellite algebra structure *(comments)*.

The slides **35**, **36** extend the main operations to satellite pairs: *lu*, *mu*, *ari*, *gari* $\to$ *bilu*, *bimu*, *biari*, *bigari*

The bimoulds $A^\bullet, B^\bullet$ on the preceding slide may be taken in $ARI^{\underline{al}/\underline{il}}_{bicoloured}$ or in $ARI^{al/.}_{bicoloured}$. See Remark 1 on slide **29****.

Similarly, on the next slide, the bimoulds $A^\bullet, B^\bullet$ may be taken in $GARI^{\underline{as}/\underline{is}}_{bicoloured}$ or in $GARI^{as/.}_{bicoloured}$.

In all cases, however, the hypothesis about the vanishing length-1 component is essential. In presence of non-vanishing length-1 components, the *satellised* operations *biari, bigari* become notably more complex: see slide **37**.

36 The satellite group structure (for symmetrals).

$$\begin{array}{ccccc} C^\bullet & \xleftarrow{\mathrm{mu}} & (A^\bullet, B^\bullet) & \xrightarrow{\mathrm{gari}} & D^\bullet \\ \mathrm{sa}\downarrow & & \mathrm{sa}\downarrow\mathrm{sa} & & \downarrow\mathrm{sa} \\ \{C_0^\bullet, C_1^\bullet\} & \xleftarrow{\mathrm{bimu}} & (\{A_0^\bullet, A_1^\bullet\}, \{B_0^\bullet, B_1^\bullet\}) & \xrightarrow{\mathrm{bigari}} & \{D_0^\bullet, D_1^\bullet\} \end{array}$$

In the absence of length-1 components:

$$\begin{aligned} \mathrm{bimu}: \quad & C_0^\bullet = \mathrm{mu}(A_0^\bullet, B_0^\bullet) \ , \quad C_1^\bullet = \mathrm{mu}(A_1^\bullet, B_1^\bullet) \\ \mathrm{bigari}: \quad & D_0^\bullet = B_0^\bullet \times \big(\mathrm{garit}(B_0^{\bullet\,-1} \times B_1^\bullet)\ A_0^\bullet\big) \\ & \quad = B_0^\bullet \times \mathrm{gari}(A_0^\bullet, B_0^{\bullet\,-1} \times B_1^\bullet) \times B_1^{\bullet\,-1} \times B_0^\bullet \\ & D_1^\bullet = B_0^\bullet \times \big(\mathrm{garit}(B_0^{\bullet\,-1} \times B_1^\bullet)\ A_1^\bullet\big) \times B_0^{\bullet\,-1} \times B_1^\bullet \\ & \quad = B_0^\bullet \times \mathrm{gari}(A_1^\bullet, B_0^{\bullet\,-1} \times B_1^\bullet) \end{aligned}$$

Remark: $D_0^{\bullet\,-1} \times D_1^\bullet \equiv \mathrm{gari}(A_0^{\bullet\,-1} \times A_1^\bullet\,,\ B_0^{\bullet\,-1} \times B_1^\bullet)$

37. Mischief potential of $\log 2$.

$$\{C_0^\bullet, C_1^\bullet\} \ \xleftarrow{\text{bilu}} \ (\{A_0^\bullet, A_1^\bullet\}, \{B_0^\bullet, B_1^\bullet\}) \ \xrightarrow{\text{biari}} \ \{D_0^\bullet, D_1^\bullet\}$$

Length-1 components (like those stemming from $\log 2$) complicate the satellite structure (see red adjuncts) :

$$\begin{aligned}
C_0^\bullet &= \mathrm{lu}(A_0^\bullet, B_0^\bullet) - \mathrm{adit}(A_1).B_0 + \mathrm{adit}(B_1).A_0 \\
C_1^\bullet &= \mathrm{lu}(A_1^\bullet, B_1^\bullet) - \mathrm{adit}(A_0).B_1 + \mathrm{adit}(B_0).A_1 \\
D_0^\bullet &= -\mathrm{ari}(A_0^\bullet, B_0^\bullet) + \mathrm{arit}(B_1^\bullet)A_0^\bullet - \mathrm{arit}(A_1^\bullet)B_0^\bullet \\
&\quad + \mathrm{adit}(A_0).B_0 - \mathrm{adit}(B_0).A_0 \\
D_1^\bullet &= +\mathrm{ari}(A_1^\bullet, B_1^\bullet) - \mathrm{arit}(B_0^\bullet)A_1^\bullet - \mathrm{arit}(A_0^\bullet)B_1^\bullet \\
&\quad - \mathrm{lu}(A_0^\bullet, B_1^\bullet) - \mathrm{lu}(A_1^\bullet, B_0^\bullet) + \mathrm{adit}(A_0).B_0 - \mathrm{adit}(B_0).A_0
\end{aligned}$$

with $C^\bullet = \mathrm{adit}(A^\bullet)B^\bullet \Leftrightarrow C^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ 0 & ,\dots, & 0 \end{smallmatrix}\right)} = \left(\sum u_i\right) A^{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)} B^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ 0 & ,\dots, & 0 \end{smallmatrix}\right)}$

37*. Mischief potential of $\log 2$ *(comments)*.

Similar, only marginally more intricate formulae account for the product *bigari* in the case of symmetral data *with* non-zero length-1 components.

This water-muddying quality of $\log 2$ (somewhat reminiscent of the nuisance potential of π^2 in the case of uncoloureds – see remark at the bottom of slide **25**) obscures the quite remarkable correspondences

$$\begin{tikzcd}[column sep=small]
\mathrm{zag}^{\binom{\mathbf{0}}{\epsilon}} \arrow[rr] \arrow[dr] & & \mathrm{zag}^{\binom{\mathbf{u}}{\mathbf{0}}} \\
& \left(\mathrm{sazag}_0^{\binom{\mathbf{u}}{\mathbf{0}}}, \mathrm{sazag}_1^{\binom{\mathbf{u}}{\mathbf{0}}}\right) \arrow[ur] &
\end{tikzcd}$$

and must be the reason why these escaped notice for so long.

38. Keeping track of the second symmetry.

The 2^{s-1} structure constraints on $ARI^{\underline{al}/\underline{il}}$ (see slide **31**):

$$\mathcal{R}^{\epsilon_1,\dots,\epsilon_{s-1}} \;:\; 0 = \sum R^{\epsilon_1,\dots,\epsilon_{s-1}}_{\epsilon'_1,\dots,\epsilon'_s} b^{\epsilon'_1,\dots,\epsilon'_s} \qquad (R^{\bullet}_{\bullet} \in \mathbb{Z})$$

respect the (r,d)-filtration: if one colour dominates in $(\epsilon_1,\dots,\epsilon_{s-1})$, it also dominates in $(\epsilon'_1,\dots,\epsilon'_s)$.

Hence two structure-and-gradation respecting isomorphisms:

$$\begin{aligned} \mathrm{ARI}^{\underline{al}/\underline{il}}_{bicoloured} &\longleftrightarrow \mathrm{BIARI}^{\underline{al}/\underline{il}^*}_{uncoloured} \\ \mathrm{GARI}^{\underline{as}/\underline{is}}_{bicoloured} &\longleftrightarrow \mathrm{BIGARI}^{\underline{as}/\underline{is}^*}_{uncoloured} \end{aligned}$$

Conjecture: The *first* ρ_s relations $\mathcal{R}^{\epsilon_1,\dots,\epsilon_{s-1}}$ imply all others, with *first* relative to the order induced by $n(\epsilon) := \sum \epsilon_i 2^i$ and $\rho_s := 1 + d_s - d_s^*$, where d_s resp. d_s^* denotes the dimension of the component of weight s in the free Lie algebra $\mathfrak{L}[e_1, e_2, e_3 \dots]$ resp. $\mathfrak{L}[e_1, e_3, e_5 \dots]$ (e_s is assigned weight s).

39. Meromorphy of $sazag_0^\bullet$ and $sazag_1^\bullet$

Despite being constructed from the $\boldsymbol{u}$-independent, 0-degree, colour-only element $zag^{\binom{0,\dots,0}{\epsilon_1,\dots,\epsilon_s}}$ of the extremal group $GARI^{\underline{as}/\underline{is}}_{extr.}$, the satellites $sazag_0^\bullet$, $sazag_1^\bullet$ retain all the essential properties of the full, $\boldsymbol{u}$-dependent $zag^\bullet$, such as

(i) meromorphy in the $\boldsymbol{u}$-variables

(ii) a modified version of the double symmetry.

Actually, the first symmetry is unchanged $(al \to al^*)$ and it is the second symmetry that undergoes a slight change: $il \to il^*$. We already (see §31) derived the *analytical* expression for il^* but we are fortunate in that il^* is also capable (like il) of a *functional* interpretation.

40. Counting our luck & listing our gains.

Our extremisation-cum-satellisation scheme succeeds only thanks to an improbable string of good luck:

Fluke 1: the restriction to the extremal algebra ($d = 0$) involves no loss of information.

Fluke 2: satellisation turns the subtractive ϵ_i-flexions into additive u_i-flexions.

Fluke 3: satellisation alters but does not destroy the second symmetry: $il \to il^*$.

Fluke 4: satellisation keeps $sazag_0^\bullet, sazag_1^\bullet$ $\boldsymbol{u}$-meromorphic.

Fluke 5: the satellisation formalism *absorbs* such key facts as
(i) the $(r, d) \leftrightarrow (d, r)$ duality for uncoloureds.
(ii) the conversion rule $zag^\bullet \leftrightarrow zig^\bullet$
(iii) the colour-consistency constraints.

41. Counting our luck & listing our gains (Cont-d)

The extremisation-cum-satellisation scheme brings huge rewards:

Gain 1: it brings about a dramatic data reduction, while allowing the algorithmic recovery of information;

Gain 2: it enables one to work entirely within the (r, d)-filtration, thereby dispelling the '*curse of imbrication*';

Gain 3: it extends *'perinomal'* irreducible analysis (*luma*$^\bullet$-based) to the coloured case;

Gain 4: it eases *'arithmetical'* irreducible analysis (*loma*$^\bullet$- or *lama*$^\bullet$-based) in all cases – uncoloured as well as coloured.

42. Concluding remarks.

• *'Arithmetical dimorphy'* extends far beyond the multizetas.
Ext.1: The $\mathbb{Q}$-ring of hyperlogarithms with rational 'support'.
Ext.2: The $\mathbb{Q}$-ring of *'naturals'* , i.e. of all monics associated with transmonomials with finitely many (rational) coefficients.

• Albeit rooted in Analysis, the *flexion structure*, with its two-tier indexation, its core involution *swap*, its wealth of operations, and its convenient capaciousness (it makes room for meromorphic functions and poles at the origin), has shown itself ideally suited to the investigation of multizeta dimorphy.

• Part 3 reflects *work in progress*: bountiful though it is, the present harvest is likely to pale before the yields of future crops...

43 Some references.

Here are two seminal papers:

[B] D.J.Broadhurst, *Conjectured Enumeration of irreducible Multiple Zeta Values, from Knots and Feynman Diagrams*, preprint, Phys. Dept, Open Univ. Milton Keynes, MK7 6AA, UK, Nov. 1996.

[Z] D.Zagier, *Values of Zeta Functions and their Applications.* First European Congress of Mathematics, Vol. 2, 427-512, Birkhäuser, Boston, 1994.

For guidance on the recent literature, look up the *Multiple Zeta Function* entry in Wikipedia.

For our own, flexion-based approach, see next page →

43* Some references.

[E1] *ARI/GARI, la dimorphie et l'arithmétique des multizetas: un premier bilan.* J.Th.N. Bordeaux, 15, 2003.
[E2] *The flexion structure and dimorphy: flexion units, singulators, generators, and the enumeration of multizeta irreducibles.* Ann.Scuo.Norm.Pisa , 2011
[E3] *Eupolars and their bialternality grid.* Acta Math.Vietnamica. 2015.
[E4] *Singulators vs Bisingulators.* 7 June 2014.
[E5] *Combinatorial tidbits from resurgence theory and mould calculus.* June 2016.

All these papers and more are accessible on the author's homepage.